

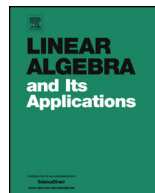


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Products of projections and self-adjoint operators [☆]



M. Laura Arias ^{a,b}, M. Celeste Gonzalez ^{c,b,*}

^a *Departamento de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Argentina*

^b *Instituto Argentino de Matemática-CONICET, Buenos Aires, Argentina*

^c *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Los Polvorines, Buenos Aires, Argentina*

ARTICLE INFO

Article history:

Received 5 December 2017

Accepted 31 May 2018

Available online 8 June 2018

Submitted by V. Muller

MSC:

47A05

47A68

47B15

Keywords:

Orthogonal projections

Self-adjoint operators

Factorization of operators

ABSTRACT

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Our goal in this article is to study the set $\mathcal{P} \cdot \mathcal{L}^h$ of operators in $\mathcal{L}(\mathcal{H})$ that can be factorized as the product of an orthogonal projection and a self-adjoint operator. We describe $\mathcal{P} \cdot \mathcal{L}^h$ and present optimal factorizations, in different senses, for an operator in this set.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. This article is devoted to the study of the set

[☆] M. Laura Arias and M. Celeste Gonzalez were supported in part by CONICET (PIP 11220120100426), FONCYT (PICT 2014-1776). M. Laura Arias was also supported by UBACYT 20020130100637.

* Corresponding author.

E-mail addresses: lauraarias@conicet.gov.ar (M. Laura Arias), celeste.gonzalez@conicet.gov.ar (M. Celeste Gonzalez).

$$\mathcal{P} \cdot \mathcal{L}^h = \{T \in \mathcal{L}(\mathcal{H}) : T = PA, P \in \mathcal{P}, A \in \mathcal{L}^h\},$$

where $\mathcal{P}$ and $\mathcal{L}^h$ denote the sets of orthogonal projections and self-adjoint operators of $\mathcal{L}(\mathcal{H})$, respectively.

Previous works on factorizations of operators in terms of particular classes of operators are in [3], [5], [7], [8] and [18] among others. In particular, the sets $\mathcal{P} \cdot \mathcal{P}$ and $\mathcal{P} \cdot \mathcal{L}^+$ where $\mathcal{L}^+$ denotes the cone of the semidefinite positive operators of $\mathcal{L}(\mathcal{H})$, are studied in [8] and [5], respectively. In these articles different characterizations of the sets $\mathcal{P} \cdot \mathcal{P}$ and $\mathcal{P} \cdot \mathcal{L}^+$ are developed and also optimal factorizations are presented. Our goal in this article is to obtain similar results for the bigger set $\mathcal{P} \cdot \mathcal{L}^h$.

Now we summarize some of the results for $\mathcal{P} \cdot \mathcal{P}$ and $\mathcal{P} \cdot \mathcal{L}^+$ than can be found in [8] and [5]. For the set $\mathcal{P} \cdot \mathcal{P}$, Crimmins (see [18, Theorem 8]) showed that $T \in \mathcal{P} \cdot \mathcal{P}$ if and only if $T^2 = TT^*T$. Later, Corach and Maestripieri in [8] showed that if $T \in \mathcal{P} \cdot \mathcal{P}$ then it can always be factorized as

$$T = P_{\overline{\mathcal{R}(T)}} P_{\mathcal{N}(T)^\perp}, \quad (1)$$

where $P_{\overline{\mathcal{R}(T)}}$ and $P_{\mathcal{N}(T)^\perp}$ denote the orthogonal projections onto the closure of the range of T and onto the orthogonal complement of the nullspace of T , respectively. They also proved that the factorization (1) is optimal in the following two senses: if $T = P_{\mathcal{M}} P_{\mathcal{N}} \in \mathcal{P} \cdot \mathcal{P}$ then

- a) $P_{\overline{\mathcal{R}(T)}} \leq P_{\mathcal{M}}$ and $P_{\mathcal{N}(T)^\perp} \leq P_{\mathcal{N}}$;
- b) $\|(P_{\overline{\mathcal{R}(T)}} - P_{\mathcal{N}(T)^\perp})x\| \leq \|(P_{\mathcal{M}} - P_{\mathcal{N}})x\|$ for all $x \in \mathcal{H}$.

On the other hand, for the set $\mathcal{P} \cdot \mathcal{L}^+$ in [5] it was proved that $T \in \mathcal{P} \cdot \mathcal{L}^+$ if and only if there exists $\lambda \geq 0$ such that $TT^* \leq \lambda TP_{\overline{\mathcal{R}(T)}}$. Furthermore, for $T \in \mathcal{P} \cdot \mathcal{L}^+$ it is always possible to find $A \in \mathcal{L}^+$ with $\mathcal{N}(A) = \mathcal{N}(T)$ such that $T = PA$, for some $P \in \mathcal{P}$. Between all elements of $\mathcal{L}^+$ with this property there exists one, denoted by $\hat{A}$, such that the factorization

$$T = P_{\overline{\mathcal{R}(T)}} \hat{A},$$

is optimal in the following senses:

- a) $P_{\overline{\mathcal{R}(T)}} \leq P$ for all $P \in \mathcal{P}$ such that $T = PA$ for some $A \in \mathcal{L}^+$;
- b) $\hat{A} \leq A$ and therefore $\|\hat{A}\| \leq \|A\|$ for all $A \in \mathcal{L}^+$ such that $T = PA$ for some $P \in \mathcal{P}$.

In this article we present general properties of operators in $\mathcal{P} \cdot \mathcal{L}^h$ and we compare the sets $\mathcal{P} \cdot \mathcal{L}^h$ and $\mathcal{P} \cdot \mathcal{L}^+$. In Section 2 we describe $\mathcal{P} \cdot \mathcal{L}^h$ and for a given $T \in \mathcal{P} \cdot \mathcal{L}^h$ we study the projection sets $\mathcal{P}_T = \{P \in \mathcal{P} : T = PA \text{ for some } A \in \mathcal{L}^h\}$ and $\mathcal{A}_T = \{A \in \mathcal{L}^h : T = PA \text{ for some } P \in \mathcal{P}\}$. Moreover, we see that given an operator $T \in \mathcal{P} \cdot \mathcal{L}^h$

it is not always possible to find $A \in \mathcal{L}^h$ with $\mathcal{N}(A) = \mathcal{N}(T)$ such that $T = PA$ for some $P \in \mathcal{P}$. We prove that this can be guaranteed under the extra hypothesis that $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. In such case, we find an element $A_{\mathcal{N}} \in \mathcal{L}^h$, with $\mathcal{N}(A_{\mathcal{N}}) = \mathcal{N}(T)$ such that the factorization

$$T = P_{\overline{\mathcal{R}(T)}} A_{\mathcal{N}},$$

is optimal in the following senses:

- a) $P_{\overline{\mathcal{R}(T)}} \leq P$ for all $P \in \mathcal{P}$ such that $T = PA$ for some $A \in \mathcal{L}^h$;
- b) $P_{\overline{\mathcal{R}(T)}} \leq^- P$ for all $P \in \mathcal{P}$ such that $T = PA$ for some $A \in \mathcal{L}^h$;
- c) $A_{\mathcal{N}} \leq^- A$, for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$.

Here, $\leq^-$ means the minus order defined for operators in $\mathcal{L}(\mathcal{H})$. Also, we distinguish another two factorizations for $T \in \mathcal{P} \cdot \mathcal{L}^h$ denoted by

$$T = P_{\overline{\mathcal{R}(T)}} A_0$$

and

$$T = P_{\overline{\mathcal{R}(T)}} A_T,$$

which are optimal in the following senses:

- a) $\|A_0\| \leq \|A\|$ for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$;
- b) $\|(T^* - A_T)x\| \leq \|(T^* - A)x\|$ for all $x \in \mathcal{H}$ and for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$;
- c) $\|T - A_T\| \leq \|T - A\|$ for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$.

See Theorems 2.2 and 3.2 for the definitions of A_T and A_0 . The results about optimal factorizations can be found in Section 3.

2. The products of projections and self-adjoint operators

We begin this section by introducing some notation. For each $X \in \mathcal{L}(\mathcal{H})$, $\mathcal{R}(X)$ and $\mathcal{N}(X)$ are the range and nullspace of X , respectively. Besides, P_X stands for the orthogonal projection from $\mathcal{H}$ onto $\overline{\mathcal{R}(X)}$. The adjoint of X is X^* and the Moore–Penrose generalized inverse of X is $X^\dagger$. We recall that $X^\dagger \in \mathcal{L}(\mathcal{H})$ if and only if X has closed range. On the other hand, if $\mathcal{V}, \mathcal{W}$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \dot{+} \mathcal{W}$ (direct sum), the symbol $Q_{\mathcal{V} // \mathcal{W}}$ identifies the oblique projection onto $\mathcal{V}$ along $\mathcal{W}$, that is, the operator $Q \in \mathcal{L}(\mathcal{H})$ with range $\mathcal{V}$ and nullspace $\mathcal{W}$ such that $Q^2 = Q$. Given $T \in \mathcal{L}(\mathcal{H})$, $T = V_T |T|$ denotes the polar decomposition of T where V_T is a partial isometry with

$\mathcal{N}(V_T) = \mathcal{N}(T)$ and $|T| = (T^*T)^{1/2}$. Finally, as we have announced in the Introduction, we shall denote by

$$\mathcal{P} \cdot \mathcal{L}^h := \{PA : P \in \mathcal{P}, A \in \mathcal{L}^h\},$$

where $\mathcal{P} := \{P \in \mathcal{L}(\mathcal{H}) : P^2 = P = P^*\}$ and $\mathcal{L}^h := \{A \in \mathcal{L}(\mathcal{H}) : A = A^*\}$.

The next result will be frequently used along the article.

Proposition 2.1. *If $T = PA \in \mathcal{P} \cdot \mathcal{L}^h$ then $T = P_TA$.*

Proof. If $T = PA$ for $P \in \mathcal{P}$ and $A \in \mathcal{L}^h$ then $\mathcal{R}(P_T) = \overline{\mathcal{R}(T)} \subseteq \mathcal{R}(P)$ and, therefore, $P_TA = P_TPA = P_TT = T$. $\square$

The following result characterizes the set $\mathcal{P} \cdot \mathcal{L}^h$. The equivalence of conditions a) and c) in the above theorem is [18, Theorem 9].

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be given. The following statements are equivalent:*

- a) $T \in \mathcal{P} \cdot \mathcal{L}^h$.
- b) $TP_T \in \mathcal{L}^h$.
- c) $T^*T^2 \in \mathcal{L}^h$.
- d) $T^n \in \mathcal{P} \cdot \mathcal{L}^h$ for all $n \in \mathbb{N}$.
- e) $|T|V_T \in \mathcal{L}^h$.
- f) $A_T = T + T^* - P_TT^* \in \mathcal{L}^h$.

Proof. a) $\leftrightarrow$ b) Assume that $T \in \mathcal{P} \cdot \mathcal{L}^h$. Then $T = P_TA$ for some $A \in \mathcal{L}^h$. So that $TP_T \in \mathcal{L}^h$. Conversely, if $TP_T \in \mathcal{L}^h$ then $A = T + T^* - TP_T \in \mathcal{L}^h$ and $T = P_TA \in \mathcal{P} \cdot \mathcal{L}^h$.

b) $\leftrightarrow$ c) Observe that $TP_T \in \mathcal{L}^h$ then for all $x, y \in \mathcal{H}$, $\langle T^*T^2x, y \rangle = \langle T^2x, Ty \rangle = \langle TP_TTx, Ty \rangle = \langle P_TT^*Tx, Ty \rangle = \langle Tx, T^2y \rangle = \langle x, T^*T^2y \rangle$, which is to say that $T^*T^2 \in \mathcal{L}^h$. Now, if $T^*T^2 = (T^2)^*T$ then by left multiplication by $(T^*)^\dagger$ and then taking adjoint we get that $(T^2)^* = T^*TP_T$. Then, again by left multiplication by $(T^*)^\dagger$ we obtain that $TP_T \in \mathcal{L}^h$.

a) $\leftrightarrow$ d) Assume that a) holds, so that $T = PA$ for some $(P, A) \in \mathcal{P} \times \mathcal{L}^h$. Pick any $k \in \mathbb{N}$. Then $T^{2k} = (PA)^{2k} = P(AP)^k(PA)^k = P(T^*)^kT^k$. On the other hand, $T^{2k+1} = TT^{2k} = PAP(T^*)^kT^k = P(T^*)^{k+1}T^k$. Note that $(T^*)^{k+1}T^k$ is self-adjoint since $(T^*)^{k+1}T^k = T^{*k}APT^k = T^{*k}AT^k$. Whence d) follows and the proof is complete.

a) $\leftrightarrow$ e) Let $T = V_T|T|$ be the polar decomposition of T . If $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $T = P_TA$ for some $A \in \mathcal{L}^h$. So that $V_T|T| = T = P_TA = V_TV_T^*A$ and therefore $V_T(V_T^*A - |T|) = 0$. Then $\mathcal{R}(V_T^*A - |T|) \subseteq \mathcal{N}(T) \cap \overline{\mathcal{R}(T^*)} = \{0\}$. Thus $V_T^*A = |T|$ and $|T|V_T$ is self-adjoint. Conversely, if $|T|V_T$ is self-adjoint then there exists $A \in \mathcal{L}^h$ such that $|T| = V_T^*A$ (see [15, Theorem 2.2] and [9, Theorem 3.5]). Then $T = V_T|T| = V_TV_T^*A = P_TA$ and the assertion follows.

a) $\leftrightarrow$ f) If $T = P_TA$, for some $A \in \mathcal{L}^h$ then $TP_T \in \mathcal{L}^h$ and so $A_T = T + T^* - TP_T \in \mathcal{L}^h$. Conversely, if $A_T \in \mathcal{L}^h$ then $P_TA_T = T \in \mathcal{P} \cdot \mathcal{L}^h$. $\square$

By the previous theorem, we get the next result concerning quasinormal operators:

Corollary 2.3. *If a quasinormal operator T (i.e., $TT^*T = T^*TT$) is in $\mathcal{P} \cdot \mathcal{L}^h$ then it is self-adjoint.*

Proof. Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ such that $TT^*T = T^*TT$. By the previous theorem, it also holds that $T^*TT = T^*T^*T$. Then, $TT^*T = T^*T^*T$ and so $TT^* = T^2$. Now, as T is quasinormal then $R(T) \subseteq \overline{R(T^*)}$ and so $T = T^*$. $\square$

Remark 2.4. *From now on, we denote by $A_T := T + T^* - TP_T$.*

Corollary 2.5.

- a) $\mathcal{P} \cdot \mathcal{L}^h$ is closed.
- b) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $T = P_TA_T$.
- c) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $T^{2k} \in \mathcal{P} \cdot \mathcal{L}^+$ for all $k \in \mathbb{N}$.
- d) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $|T|V_T \in \mathcal{L}^+$ then $T^{2k+1} \in \mathcal{P} \cdot \mathcal{L}^+$ for all $k \in \mathbb{N}$.

Proof. a) It follows from item c) of Theorem 2.2.

b) It follows from the proof of a) $\rightarrow$ d) of Theorem 2.2.

c) From a) $\rightarrow$ b) of Theorem 2.2 we know that $T^{2k} = P_T(T^*)^kT^*$. Then $T \in \mathcal{P} \cdot \mathcal{L}^+$.

d) Since $|T|V_T \in \mathcal{L}^+$ then $(T^*)^2T = T^*|T|^2 = |T|V_T^*|T||T| \in \mathcal{L}^+$. Now, as $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $T^{2k+1} \in \mathcal{P} \cdot \mathcal{L}^h$ for all $k \in \mathbb{N}$. From the proof of Theorem 2.2 we get that $T^{2k+1} = P(T^*)^{k+1}T^k$. Observe that $(T^*)^{k+1}T^k = (T^*)^{k-1}(T^*)^2TT^{k-1} \in \mathcal{L}^+$. So that, $T^{2k+1} \in \mathcal{P} \cdot \mathcal{L}^+$. $\square$

Remark 2.6. *Observe that given $T \in \mathcal{P} \cdot \mathcal{L}^h$, T^{2k+1} is not necessarily in $\mathcal{P} \cdot \mathcal{L}^+$ for all $k \in \mathbb{N}$. For example, consider $T = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{L}^h$. Note that $T^{2k+1} = T$ for all $k \in \mathbb{N}$. However, since $TP_T = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \notin \mathcal{L}^+$ then, by [5, Theorem 3.2], $T^{2k+1} \notin \mathcal{P} \cdot \mathcal{L}^+$.*

Remark 2.7. *The following example shows that:*

- a) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $\overline{\mathcal{R}(T)} \cap \mathcal{N}(T) \neq \{0\}$, in general;
- b) $\mathcal{P} \cdot \mathcal{L}^+ \subsetneq \mathcal{P} \cdot \mathcal{L}^h$;
- c) $\mathcal{P} \cdot \mathcal{L}^h$ is not closed by adjunction.

In fact, consider $T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. Since $T^*T^2 \in \mathcal{L}^h$ then $T \in \mathcal{P} \cdot \mathcal{L}^h$. Then $\mathcal{R}(T) \cap \mathcal{N}(T) \neq \{0\}$. On the other hand, note that $\mathcal{P} \cdot \mathcal{L}^+ \subsetneq \mathcal{P} \cdot \mathcal{L}^h$ because if $T \in \mathcal{P} \cdot \mathcal{L}^+$ then $\overline{\mathcal{R}(T)} \cap \mathcal{N}(T) = \{0\}$ (see [5, Lemma 3.1]). Finally, to see that $\mathcal{P} \cdot \mathcal{L}^h$ is not closed by adjunction, it is sufficient to check that $T(T^*)^2 \notin \mathcal{L}^h$.

From now on, given $T \in \mathcal{P} \cdot \mathcal{L}^h$, we set

$$\mathcal{P}_T := \{P \in \mathcal{P} : PA = T \text{ for some } A \in \mathcal{L}^h\}$$

and

$$\mathcal{A}_T := \{A \in \mathcal{L}^h : PA = T \text{ for some } P \in \mathcal{P}\}.$$

In the next two results, we study the projection sets $\mathcal{P}_T$ and $\mathcal{A}_T$.

Proposition 2.8. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $P \in \mathcal{P}$. The following assertions are equivalent:*

- a) $P \in \mathcal{P}_T$.
- b) $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and $TP = PT^*$.
- c) $PA_T = T$.

Proof. a) $\rightarrow$ b) Suppose that there exists $A \in \mathcal{L}^h$ such that $PA = T$. Then $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and $TP = PAP = PT^*$.

b) $\rightarrow$ c) If $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and $TP = PT^*$ then $PA_T = PT + PT^* - PP_T T^* = PT = T$.

c) $\rightarrow$ a) Since $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $A_T \in \mathcal{L}^h$ and so $P \in \mathcal{P}_T$. $\square$

Proposition 2.9. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $A \in \mathcal{L}^h$. The following assertions are equivalent:*

- a) $A \in \mathcal{A}_T$.
- b) $P_T A = T$.
- c) $A = A_T + X$ for some $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$.

Proof. a) $\rightarrow$ b) If $PA = T$ for some $P \in \mathcal{P}$, then $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and, whence, $T = P_T A$.

b) $\rightarrow$ c) Note that $A = P_T A P_T + P_T A (I - P_T) + (I - P_T) A P_T + (I - P_T) A (I - P_T) = TP_T + T(I - P_T) + (I - P_T)T^* + (I - P_T)A(I - P_T) = T + T^* - P_T T^* + (I - P_T)A(I - P_T) = A_T + (I - P_T)A(I - P_T)$ and the assertion follows.

c) $\rightarrow$ a) Since $P_T A = P_T A_T = T$ then $A \in \mathcal{A}_T$. $\square$

Proposition 2.10. *The set $\mathcal{A}_T$ is a closed (in norm) $\mathbb{R}$ -affine manifold.*

Proof. Item c) of Proposition 2.9 shows that $\mathcal{A}_T$ is a $\mathbb{R}$ -affine manifold. Now let us see that $\mathcal{A}_T$ is closed. If $\{A_n\} \subseteq \mathcal{A}_T$ and $A_n \xrightarrow[n \rightarrow \infty]{\|\cdot\|} A$ then $A \in \mathcal{L}^h$ and $A_n x \xrightarrow[n \rightarrow \infty]{} Ax$ for all $x \in \mathcal{H}$. Then $Tx = P_T A_n x \xrightarrow[n \rightarrow \infty]{} P_T Ax$. So that $P_T A = T$ and therefore $A \in \mathcal{A}_T$. $\square$

In [5, Proposition 4.1] it was proved that if $T \in \mathcal{P} \cdot \mathcal{L}^+$ then it always exists $A \in \mathcal{L}^+$ such that $T = P_T A$ and $\mathcal{N}(A) = \mathcal{N}(T)$. Furthermore, it was shown that this special factor in $\mathcal{L}^+$ turns to have optimal properties among all $A \in \mathcal{L}^+$ such that $T = PA$ for some $P \in \mathcal{P}$. Motivated by this, given $T \in \mathcal{P} \cdot \mathcal{L}^h$ we are interested in finding $A \in \mathcal{A}_T$ such that $\mathcal{N}(A) = \mathcal{N}(T)$. Unfortunately, it is not always possible in $\mathcal{P} \cdot \mathcal{L}^h$. For instance, consider $T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. By Remark 2.7 it holds that $T \in \mathcal{P} \cdot \mathcal{L}^h \setminus \mathcal{P} \cdot \mathcal{L}^+$.

It is easy to check that $A_T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{pmatrix}$. Then, by Proposition 2.9, every $A \in \mathcal{A}_T$ is $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & x \end{pmatrix}$, with $x \in \mathbb{R}$. Since $\det(A) \neq 0$ for all $x \in \mathbb{R}$ then A is invertible. Therefore, $\mathcal{N}(A) \neq \mathcal{N}(T)$ for all $A \in \mathcal{A}_T$.

The next result will be useful in order to study when it is possible to find an $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$.

Proposition 2.11. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $A \in \mathcal{A}_T$. The following statements hold:*

- a) $\overline{\mathcal{R}(T)} \cap \mathcal{N}(A) = \{0\}$ (and therefore $\mathcal{R}(A) + \mathcal{R}(T)^\perp$ is dense in $\mathcal{H}$);
- b) T has closed range if and only if $\mathcal{H} = \mathcal{R}(A) + \mathcal{R}(T)^\perp$;
- c) $\mathcal{R}(T)^\perp \cap \mathcal{R}(A) = \{0\}$ if and only if $\mathcal{N}(A) = \mathcal{N}(T)$.

Proof. a) Take $x \in \overline{\mathcal{R}(T)} \cap \mathcal{N}(A)$. Then $x = P_T x$ and $0 = Ax = AP_T x = T^* x$. So that $x \in \overline{\mathcal{R}(T)} \cap \mathcal{N}(T^*) = \{0\}$.

b) First, let us see that $\mathcal{R}(T) \subseteq \mathcal{R}(A) + \mathcal{R}(T)^\perp$. In fact, if $y \in \mathcal{R}(T)$ then $y = Tx = P_T Ax$ for some $x \in \mathcal{H}$. Then $P_T Tx = P_T Ax$ and so $Tx - Ax \in \mathcal{R}(T)^\perp$. Therefore the inclusion is obtained. Now, if T has closed range then $\mathcal{H} = \mathcal{R}(T) + \mathcal{R}(T)^\perp \subseteq \mathcal{R}(A) + \mathcal{R}(T)^\perp$. Conversely, suppose that $\mathcal{H} = \mathcal{R}(A) + \mathcal{R}(T)^\perp$ and $T = P_T A$. Hence, $\overline{\mathcal{R}(T)} = P_T(\mathcal{H}) = P_T(\mathcal{R}(A) + \mathcal{R}(T)^\perp) = \mathcal{R}(P_T A) = \mathcal{R}(T)$, i.e., T has closed range.

c) Let $T = P_T A$. Suppose that $\mathcal{N}(A) = \mathcal{N}(T)$. If $y \in \mathcal{R}(A) \cap \mathcal{R}(T)^\perp$ then $y = Ax$ for some $x \in \mathcal{H}$ and $P_T y = 0$. So that, $0 = P_T Ax = Tx$. Hence $x \in \mathcal{N}(T) = \mathcal{N}(A)$ and, therefore $y = 0$. Conversely, since $T = P_T A$ it is clear that $\mathcal{N}(A) \subseteq \mathcal{N}(T)$. Let $x \in \mathcal{N}(T)$. Then $0 = P_T Ax$ and so $Ax \in \mathcal{R}(A) \cap \mathcal{R}(T)^\perp = \{0\}$. Then $x \in \mathcal{N}(A)$ and then the assertion follows. $\square$

Theorem 2.12. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$. If there exists $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$ then $\overline{\mathcal{R}(T)} + \mathcal{N}(T)$ is dense in $\mathcal{H}$. Conversely, if $\overline{\mathcal{R}(T)} + \mathcal{N}(T) = \mathcal{H}$ then there exists $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$.*

Proof. Let $T \in \mathcal{P} \cdot \mathcal{L}^h$. Assume that $T = P_T A$ with $A \in \mathcal{L}^h$ and $\mathcal{N}(A) = \mathcal{N}(T)$. Then, by items a) and c) of Proposition 2.11, it holds that $\overline{\mathcal{R}(T)} \cap \mathcal{N}(T) = \{0\}$ and $\mathcal{N}(A) + \overline{\mathcal{R}(T)}$ is dense in $\mathcal{H}$. Therefore $\overline{\mathcal{R}(T)} + \mathcal{N}(T)$ is dense in $\mathcal{H}$. On the other hand,

if $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$, take $Q = Q_{\overline{\mathcal{R}(T)}/\mathcal{N}(T)}$ and define $A := A_T Q$. Note that $A = T^*Q = Q^*P_T T^*Q \in \mathcal{L}^h$ because of Theorem 2.2. Furthermore $\mathcal{N}(T) \subseteq \mathcal{N}(A)$ and if $Ax = A_T Qx = 0$ then $Qx \in \mathcal{N}(A_T) \cap \overline{\mathcal{R}(T)} = \{0\}$ (see Proposition 2.11). Then $x \in \mathcal{N}(Q) = \mathcal{N}(T)$ and so $\mathcal{N}(A) = \mathcal{N}(T)$. In addition $P_TA = P_TA_T Q = TQ = T$. The proof is complete. $\square$

Corollary 2.13. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ with closed range. The next conditions are equivalent:*

- a) *There exists $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$.*
- b) *$\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$.*

Proof. Assume that there exists $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$. Then, by Theorem 2.12, $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$ is dense in $\mathcal{H}$. We claim that $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$ is closed. In fact, by Proposition 2.11, as T has closed range then $\mathcal{H} = \mathcal{R}(A) \dot{+} \mathcal{R}(T)^\perp$ and so $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$ as desired. The converse follows by Theorem 2.12. $\square$

In the next proposition, given $T \in \mathcal{P} \cdot \mathcal{L}^h$ we present equivalent conditions to those of Theorem 2.12 in order to guarantee the existence of an $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$. For that, given a pair $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$ of closed subspaces we shall denote by $c_0(\mathcal{V}, \mathcal{W})$ to the cosine of the Dixmier angle between $\mathcal{V}$ and $\mathcal{W}$, i.e.,

$$c_0(\mathcal{V}, \mathcal{W}) := \sup\{|\langle v, w \rangle| : v \in \mathcal{V}, w \in \mathcal{W}, \|v\| = \|w\| = 1\}.$$

It holds that $c_0(\mathcal{V}, \mathcal{W}) < 1$ if and only if $\mathcal{V} + \mathcal{W}$ is closed and $\mathcal{V} \cap \mathcal{W} = \{0\}$ (see [11, Theorem 1]).

Proposition 2.14. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$. The following conditions are equivalent:*

- a) $c_0\left(\mathcal{R}(T)^\perp, \overline{A_T(\mathcal{R}(T))}\right) < 1$;
- b) $c_0\left(\mathcal{N}(P), \overline{A(\mathcal{R}(P))}\right) < 1$ for all $P \in \mathcal{P}, A \in \mathcal{L}^h$ such that $T = PA$;
- c) $\mathcal{H} = \overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$.

Proof. a) $\rightarrow$ b) By Proposition 2.9 every $A \in \mathcal{A}_T$ can be written as $A = A_T + X$ for some $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$. Now, if $P \in \mathcal{P}_T$ then $\mathcal{N}(P) \subseteq \mathcal{R}(T)^\perp$. Furthermore $\overline{A(\mathcal{R}(P))} = \overline{\mathcal{R}(T^*)}$ and $\overline{A_T \mathcal{R}(T)} = \overline{A_T \mathcal{R}(P_T)} = \overline{\mathcal{R}(T^*)}$. Thus the assertion follows since $c_0\left(\mathcal{N}(P), \overline{A(\mathcal{R}(P))}\right) \leq c_0\left(\mathcal{R}(T)^\perp, \overline{A_T \mathcal{R}(T)}\right)$.

b) $\rightarrow$ c) Since $c_0\left(\mathcal{N}(P), \overline{A(\mathcal{R}(P))}\right) < 1$ for all $P \in \mathcal{P}$ and $A \in \mathcal{L}^h$ such that $T = PA$ then, in particular, $c_0\left(\mathcal{R}(T)^\perp, \overline{A_T(\mathcal{R}(T))}\right) < 1$. Now observe that $\overline{A_T(\mathcal{R}(T))} = \overline{\mathcal{R}(T^*)}$. Then we get that $\mathcal{R}(T)^\perp \dot{+} \overline{\mathcal{R}(T^*)}$ is closed. In consequence, $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. In addition, if $x \in \overline{\mathcal{R}(T)} \cap \mathcal{N}(T)$ then $x = P_T x$ and $0 = TP_T x = P_T T^* x$. So that $T^* x \in \mathcal{R}(T^*) \cap \mathcal{R}(T)^\perp \subseteq \overline{\mathcal{R}(T^*)} \cap \mathcal{R}(T)^\perp = \{0\}$. Therefore $x \in \mathcal{N}(T^*) \cap \overline{\mathcal{R}(T)} = \{0\}$ as desired.

c) $\rightarrow$ a) Since $\mathcal{N}(T)^\perp = \overline{A_T(\overline{\mathcal{R}(T)})}$ then the assertion follows by [11, Theorem 12 and Theorem 16]. $\square$

Remark 2.15. Given a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ and $A \in \mathcal{L}^h$, it is said that the pair $(A, \mathcal{S})$ is compatible if there exists $Q \in \mathcal{L}(\mathcal{H})$ such that $Q^2 = Q$, $\mathcal{R}(Q) = \mathcal{S}$ and $AQ = Q^*A$. This notion was introduced and studied in [16]. It was proved that the pair $(A, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}^\perp, \overline{A(\mathcal{S})}) < 1$ ([16, Theorem 4.7]). Therefore, observe that given $T \in \mathcal{P} \cdot \mathcal{L}^h$, the conditions of Proposition 2.14 are equivalent to the compatibility of the pair $(A_T, \overline{\mathcal{R}(T)})$ and also to the compatibility of the pair $(A, \mathcal{R}(P))$ for all $A \in \mathcal{L}^h$ and $P \in \mathcal{P}$ such that $T = PA$.

For the next result we denote by $\mathcal{I}_0$ the set of split partial isometries of $\mathcal{L}(\mathcal{H})$, i.e., the set of partial isometries V such that $\mathcal{R}(V) \dot{+} \mathcal{N}(V) = \mathcal{H}$. This class of operators was studied in [1].

Proposition 2.16. Let $T \in \mathcal{L}(\mathcal{H})$. The following assertions are equivalent:

- a) $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$;
- b) $|T|_{V_T} \in \mathcal{L}^h$ and $V_T \in \mathcal{I}_0$.

Proof. The proof follows from Theorem 2.2 and the facts that $\mathcal{R}(V_T) = \overline{\mathcal{R}(T)}$ and $\mathcal{N}(V_T) = \mathcal{N}(T)$. $\square$

Definition 1. Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ be such that $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. If $Q = Q_{\overline{\mathcal{R}(T)}/\mathcal{N}(T)}$ we define

$$A_{\mathcal{N}} = A_T Q.$$

Observe that, by the proof of Theorem 2.12, $A_{\mathcal{N}} \in \mathcal{A}_T$ and $\mathcal{N}(A) = \mathcal{N}(T)$.

Proposition 2.17. The operator $A_{\mathcal{N}}$ satisfies the following properties:

- a) $A_{\mathcal{N}}$ is the unique operator in $\mathcal{A}_T$ with nullspace equal to $\mathcal{N}(T)$.
- b) $\mathcal{R}(A_{\mathcal{N}})$ is closed if and only if $\mathcal{R}(T)$ is closed.

Proof. a). Suppose that there exists $A \in \mathcal{A}_T$ such that $\mathcal{N}(A) = \mathcal{N}(A_{\mathcal{N}}) = \mathcal{N}(T)$. Then $\mathcal{R}(A - A_{\mathcal{N}}) \subseteq \mathcal{N}(T^*)$ since $T^*(A - A_{\mathcal{N}}) = AP_T(A - A_{\mathcal{N}}) = A(T - T) = 0$. On the other hand, as $\mathcal{N}(A) = \mathcal{N}(A_{\mathcal{N}}) = \mathcal{N}(T)$ then $\mathcal{R}(A - A_{\mathcal{N}}) \subseteq \mathcal{N}(T)^\perp$. Hence, $\mathcal{R}(A - A_{\mathcal{N}}) \subseteq \mathcal{N}(T^*) \cap \mathcal{N}(T)^\perp = \{0\}$ because $\mathcal{H} = \overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$, so $A = A_{\mathcal{N}}$.

b) Suppose that $\mathcal{R}(A_{\mathcal{N}})$ is closed. Then, $\mathcal{R}(A_{\mathcal{N}}) = \overline{\mathcal{R}(T^*)}$ and so, $\mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{N}(T^*) = \mathcal{H}$ because $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. Therefore, by Proposition 2.11, $\mathcal{R}(T)$ is closed.

Conversely, if $\mathcal{R}(T)$ is closed then, by Proposition 2.11, $\mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(T)^\perp = \mathcal{H}$. Hence, applying [13, Theorem 2.3], we obtain that $\mathcal{R}(A_{\mathcal{N}})$ is closed. $\square$

Remark 2.18. Notice that if $T \in \mathcal{P} \cdot \mathcal{L}^+$ with $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ then $A_{\mathcal{N}}$ coincides with the optimal operator in $\mathcal{L}^+$ given in [5, Remark 4.2]. In fact, by [5, Proposition 4.1], there exists a unique $A \in \mathcal{L}^+$ with $\mathcal{N}(A) = \mathcal{N}(T)$ such that $T = P_T A$. Therefore, it is sufficient to show that $A_{\mathcal{N}} \in \mathcal{L}^+$. Now, $A_{\mathcal{N}} = A_T Q = T^* Q = Q^* T^* Q = Q^* P_T T^* Q \in \mathcal{L}^+$ because by [5, Theorem 3.2], $P_T T^* \in \mathcal{L}^+$.

Proposition 2.19. Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ with $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. Then the following assertions hold:

- a) For every $A \in \mathcal{A}_T$ there exists $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$ such that $A = A_{\mathcal{N}} + X$. Furthermore $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(X)$.
- b) There exists $A \in \mathcal{A}_T$ with dense range.
- c) There exists $A \in \mathcal{A}_T$ invertible if and only if $\mathcal{R}(T)$ is closed.

Proof. a) It is easy to check that every $A \in \mathcal{A}_T$ can be written as $A = A_{\mathcal{N}} + X$, for some $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$. Now, since $\overline{\mathcal{R}(A_{\mathcal{N}})} \dot{+} \overline{\mathcal{R}(X)} = \overline{\mathcal{R}(T^*)} \dot{+} \overline{\mathcal{R}(X)}$ is closed then, by [6, Theorem 3.10], we get that $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(X)$.

b) Define $A = A_{\mathcal{N}} + (I - P_T)$. By the above item $A \in \mathcal{A}_T$ and, since $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) + \mathcal{N}(T^*)$ and $\overline{\mathcal{R}(A_{\mathcal{N}})} = \overline{\mathcal{R}(T^*)}$ it holds that A has dense range.

c) If there exists $A \in \mathcal{A}_T$ invertible then $\mathcal{R}(T) = \mathcal{R}(P_T A) = \mathcal{R}(P_T) = \overline{\mathcal{R}(T)}$. So that T has closed range. Conversely, if $\mathcal{R}(T)$ is closed then $A = A_{\mathcal{N}} + (I - P_T) \in \mathcal{A}_T$ and $\mathcal{R}(A) = \mathcal{H}$. Therefore, A is invertible. $\square$

Proposition 2.20. Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ with closed range such that $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$. Then the following assertions hold:

- a) $Q_{\mathcal{R}(T)/\mathcal{N}(T)} = (A_{\mathcal{N}} P_T)^\dagger A_{\mathcal{N}} = (T^*)^\dagger A_{\mathcal{N}}$;
- b) $\{A \in \mathcal{A}_T : \mathcal{R}(A) \text{ is closed}\} = \{A_{\mathcal{N}} + X : X \in \mathcal{L}^h, \mathcal{R}(X) \text{ is closed and } \mathcal{R}(X) \subseteq \mathcal{N}(T^*)\}$;
- c) $T^\dagger \in \mathcal{P} \cdot \mathcal{L}^h$.

Proof. a) This proof is similar to the proof of [5, Proposition 4.3].

b) It is clear that every $A \in \mathcal{A}_T$ can be written as $A = A_{\mathcal{N}} + X$, for some $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$. Since $\mathcal{H} = \mathcal{R}(T) \dot{+} \mathcal{N}(T)$ then $\mathcal{H} = \mathcal{R}(T^*) \dot{+} \mathcal{N}(T^*)$. So, $c_0(\mathcal{R}(A_{\mathcal{N}}), \overline{\mathcal{R}(X)}) \leq c_0(\mathcal{R}(T^*), \mathcal{N}(T^*)) < 1$. Thus $\mathcal{R}(A_{\mathcal{N}}) \dot{+} \overline{\mathcal{R}(X)}$ is closed. Then by [6, Theorem 3.10] it holds that $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(X)$. Therefore it is clear that if $\mathcal{R}(X)$ is closed then $\mathcal{R}(A)$ is closed. Conversely, if $\mathcal{R}(A)$ is closed then by [13, Theorem 2.3] it holds that $\mathcal{R}(X)$ is closed.

c) By Proposition 2.19 there exists $A \in \mathcal{L}^h$ invertible such that $P_T A = T$. Define $C := P_{R(AP)} A^{-1} \in \mathcal{P} \cdot \mathcal{L}^h$. Therefore it holds that C has closed range, $TC = P_T$ and $\mathcal{R}(C) \subseteq \mathcal{N}(T)^\perp$. Thus, by [4, Theorem 3.1], $C = T^\dagger$ and so $T^\dagger \in \mathcal{P} \cdot \mathcal{L}^h$. $\square$

3. Optimal decompositions

This section is devoted to the study of optimal factors in $\mathcal{P}_T$ and $\mathcal{A}_T$ for $T \in \mathcal{P} \cdot \mathcal{L}^h$. We shall consider three different criteria of optimality: minimization with respect to usual order between self-adjoint operators, minimization with respect to the minus order in $\mathcal{L}(\mathcal{H})$ and minimization of the distance to T . By usual order between selfadjoint operators we mean that given $A, B \in \mathcal{L}^h$, $A \leq B$ if $B - A \in \mathcal{L}^+$. For the minus order we shall use the symbol $\leq^-$. Given $A, B \in \mathcal{L}(\mathcal{H})$, it is said that $A \leq^- B$ if and only if there exist two idempotents Q_1 and Q_2 in $\mathcal{L}(\mathcal{H})$ such that $A = Q_1 B$ and $A^* = Q_2 B^*$. The minus order was introduced by Hartwig [14] and independently by Nambooripad [17] on semigroups. Later this order was extended to operators in $\mathcal{L}(\mathcal{H})$ by Antezana, Corach and Stojanoff [2] and by Šmerl [19].

Let us start studying the optimality in $\mathcal{P}_T$:

Proposition 3.1. *If $T \in \mathcal{P} \cdot \mathcal{L}^h$ then:*

- a) $P_T = \min\{P : P \in \mathcal{P}_T\}$, where the minimum is taken with respect usual order between self-adjoint operators.
- b) $P_T = \min\{P : P \in \mathcal{P}_T\}$, where the minimum is taken with respect to the minus order.

Proof. Let $P \in \mathcal{P}_T$. Then $\overline{\mathcal{R}(T)} \subseteq \mathcal{R}(P)$. So that, is clear that $P_T \leq P$. Furthermore, $P_T = P_T P$. Then $P_T \leq^- P$. $\square$

In [5, Proposition 4.7] it was proven that if $T \in \mathcal{P} \cdot \mathcal{L}^+$ then there exists $\hat{A} \in \mathcal{L}^+$ with $\mathcal{N}(\hat{A}) = \mathcal{N}(T)$ and $T = P_T \hat{A}$ such that $\hat{A}$ realizes the minimum among all the positive operators A such that $T = P_T A$ in two ways: with respect to the operator norm and with respect to the usual order defined on the set of self-adjoint operators. Hence, one may wonder if a similar result can be obtained for $T \in \mathcal{P} \cdot \mathcal{L}^h$. But, as the next example shows, it is not possible, in general. For example, consider $T = PA = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{L}^h$. It is easy to check that $A_{\mathcal{N}} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}^h$. It holds that, by Proposition 2.17 we know that $A_{\mathcal{N}}$ is the unique operator in $\mathcal{A}_T$ with nullspace $\mathcal{N}(T)$. But, $\|A_{\mathcal{N}}\| = 2 \geq \sqrt{2} = \|T\|$. However, as we will see in the next result, the set $\mathcal{A}_T$ has a minimum with respect to the operator norm and the minimal norm of the operators in this set coincides with the norm of operator T . We include its proof for the sake of completeness. However, the arguments are very similar to those in [10, Section 1] where the problem of finding the entry D in the block operator matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ so as to satisfy the norm bound $\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\| \leq \mu$, for given Hilbert space operators A, B, C and prescribed μ , is fully studied.

Theorem 3.2. *Given $T \in \mathcal{P} \cdot \mathcal{L}^h$ it holds that*

$$\min_{A \in \mathcal{A}_T} \|A\| = \|T\|.$$

Moreover, the minimum is achieved in the operator A_0 defined in (3).

Proof. Write

$$T_1 := T|_{\overline{\mathcal{R}(T)}} \quad \text{and} \quad T_2 := T|_{\mathcal{N}(T^*)}.$$

For all $h \in \mathcal{H}$

$$\|T\|^2 \|P_T h\|^2 = \|T^*\|^2 \|P_T h\|^2 \geq \|T^* P_T h\|^2 = \|T_1 P_T h\|^2 + \|T_2^* P_T h\|^2$$

whence

$$\langle T_2 T_2^* P_T h, P_T h \rangle \leq \langle (\|T\|^2 - T_1^2) P_T h, P_T h \rangle. \quad (2)$$

Put $\alpha := \|T\|$,

$$D_\alpha := (\alpha^2 |_{\overline{\mathcal{R}(T)}} - T_1^2)^{\frac{1}{2}} \quad \text{and} \quad \mathcal{D}_\alpha := \overline{\mathcal{R}(D_\alpha)}.$$

Then (2) yields a contraction $C_\alpha : \mathcal{D}_\alpha \rightarrow \mathcal{N}(T^*)$ such that $T_2^* = C_\alpha D_\alpha$. In particular, $A_T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & 0 \end{pmatrix}$ can be written as

$$A_T = \begin{pmatrix} T_1 & D_\alpha C_\alpha^* \\ C_\alpha D_\alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha \end{pmatrix} \begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha^* \end{pmatrix}.$$

Take $X_0 := -C_\alpha T_1 C_\alpha^* \in \mathcal{L}^h(\mathcal{N}(T^*))$ and $A_0 := A_T + X_0 \in \mathcal{A}_T$, so that

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha \end{pmatrix} \begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & -T_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha^* \end{pmatrix}. \quad (3)$$

It is well known that the block operator matrix $\begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & -T_1 \end{pmatrix}$ is α times a unitary operator on $\overline{\mathcal{R}(T)} \oplus \mathcal{D}_\alpha$. Thus, for all $h \in \overline{\mathcal{R}(T)}$ and $x \in \mathcal{D}_\alpha$,

$$\left\| \begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & -T_1 \end{pmatrix} \begin{pmatrix} h \\ x \end{pmatrix} \right\| = \alpha \left\| \begin{pmatrix} h \\ x \end{pmatrix} \right\|.$$

Therefore, $\|A_0\| \leq \alpha = \|T\|$. Indeed, as $\|T\| \leq \|A\|$, for all $A \in \mathcal{A}_T$, it turns out that $\|A_0\| = \|T\| = \min_{A \in \mathcal{A}_T} \|A\|$. $\square$

Note that the operator A_T does not realize the minimum in Theorem 3.2. In fact, consider $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{L}^h$. Here, $A_T = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$ and $\|A_T\| = \frac{1+\sqrt{6}}{2} > \sqrt{2} = \|T\|$. However A_T is optimal in the next sense:

Theorem 3.3. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$. Then*

$$\min_{A \in \mathcal{A}_T} \|(T^* - A)x\| = \|(T^* - A_T)x\| \text{ for all } x \in \mathcal{H}. \quad (4)$$

Moreover A_T is the unique operator in $\mathcal{A}_T$ which realizes the minimum in (4). In particular, it holds that

$$\min_{A \in \mathcal{A}_T} \|T - A\| = \|T - A_T\|, \quad (5)$$

Proof. Let $x \in \mathcal{H}$ and $A \in \mathcal{A}_T$. Then $\|(T^* - A)x\|^2 = \|(T^* - A_T - X)x\|^2 = \|(T^* - T - (I - P_T)T^* - X)x\|^2 = \|(P_T T^* - T - X)x\|^2 = \|(TP_T - T - X)x\|^2 = \|T(P_T - I)x\|^2 + \|Xx\|^2 \geq \|T(P_T - I)x\|^2 = \|(T^* - A_T)x\|^2$. In addition, if there exists another $A_1 = A_T + X_1 \in \mathcal{A}_T$ such that $\|(T^* - A_1)x\| \leq \|(T^* - A_T)x\|$ for all $x \in \mathcal{H}$ then, in particular, $\|(T^* - A_1)x\| \leq \|(T^* - A_T)x\|$ for all $x \in \mathcal{H}$. Hence $\|X_1x\| = 0$ for all $x \in \mathcal{H}$. So that $X_1 = 0$ and therefore $A_1 = A_T$. Finally, from the above we get that $\|T - A_T\| = \|T^* - A_T\| \leq \|T^* - A\| = \|T - A\|$. $\square$

Finally, given $T \in \mathcal{P} \cdot \mathcal{L}^h$ with $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ we shall prove that $A_{\mathcal{N}}$ is optimal in $\mathcal{A}_T$ with respect to the minus order in $\mathcal{L}(\mathcal{H})$. For this we use the following result due to Dijić, Fongi and Maestripieri [12, Proposition 3.2].

Proposition 3.4. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The following assertions are equivalent:*

- a) $A \leq^- B$;
- b) $\mathcal{N}(A) + \mathcal{N}(B - A) = \mathcal{N}(A^*) + \mathcal{N}(B^* - A^*) = \mathcal{H}$.

Theorem 3.5. *If $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ then*

$$A_{\mathcal{N}} = \min\{A : A \in \mathcal{A}_T\},$$

where the minimum is taken with respect to the minus order. Moreover, $A_{\mathcal{N}}$ is the unique element in $\mathcal{A}_T$ that realizes the minimum.

Proof. By Proposition 2.19 every $A \in \mathcal{A}_T$ can be written as $A = A_{\mathcal{N}} + X$, for some $X = X^*$ and $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$. Furthermore $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) + \mathcal{R}(X)$. Now, $\mathcal{H} = \overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) \subseteq \mathcal{N}(A - A_{\mathcal{N}}) + \mathcal{N}(A_{\mathcal{N}})$. Then, by Proposition 3.4, we get that $A_{\mathcal{N}} \leq^- A$. Now, suppose that there exists $\tilde{A} \in \mathcal{A}_T$ such that $\tilde{A} \leq^- A$ for all $A \in \mathcal{A}_T$. In particular it holds that $\tilde{A} \leq A_{\mathcal{N}}$. Then there exists an idempotent $Q \in \mathcal{L}(\mathcal{H})$ such that

$\tilde{A} = QA_N$. Then $\mathcal{N}(T) = \mathcal{N}(A_N) \subseteq \mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(T)$. Thus $\mathcal{N}(\tilde{A}) = \mathcal{N}(T)$ and therefore, by Proposition 2.17, $\tilde{A} = A_N$. $\square$

Acknowledgements

We would like to thank Professor Gustavo Corach for suggesting us to study the set $\mathcal{P} \cdot \mathcal{L}^h$ and for the carefully reading of this manuscript. In addition, we would like to thank Professor Stefania Marcantognini for showing us Theorem 3.2 in a private communication and for allowing us to add it to this article. Finally, we thank the referee for the carefully reading of this article and for the comment concerning quasinormal operators which we include as Corollary 2.3.

References

- [1] E. Andruchow, G. Corach, M. Mbekhta, Split partial isometries, *Complex Anal. Oper. Theory* 7 (2013) 813–829.
- [2] J. Antezana, G. Corach, D. Stojanoff, Bilateral shorted operators and parallel sums, *Linear Algebra Appl.* 414 (2–3) (2006) 570–588.
- [3] M. Laura Arias, Jorge Antezana, Gustavo Corach, On some factorizations of operators, *Linear Algebra Appl.* 515 (2017) 226–245.
- [4] M. Laura Arias, Gustavo Corach, M. Celeste Gonzalez, Generalized inverses and Douglas equations, *Proc. Amer. Math. Soc.* 136 (2008) 3177–3183.
- [5] M. Laura Arias, Gustavo Corach, M. Celeste Gonzalez, Products of projections and positive operators, *Linear Algebra Appl.* 439 (2013) 1730–1741.
- [6] M.L. Arias, G. Corach, M.C. Gonzalez, Additivity properties of operator ranges, *Linear Algebra Appl.* 439 (2013) 3581–3590.
- [7] M.L. Arias, G. Corach, A. Maestripieri, Products of idempotent operators, *Integral Equations Operator Theory* 88 (2017) 269–286.
- [8] G. Corach, A. Maestripieri, Products of orthogonal projections and polar decompositions, *Linear Algebra Appl.* 434 (2011) 1594–1609.
- [9] A. Dajić, J. Koliha, Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators, *Linear Algebra Appl.* 429 (2008) 1779–1809.
- [10] Chandler Davis, Dilations preserving a bound on the norm of an operator, in: B. Jefferies, A. McIntosh (Eds.), *Miniconference on Operator Theory and Partial Differential Equations*, Canberra, 1984, pp. 3–14.
- [11] F. Deutsch, The angles between subspaces of a Hilbert space, in: S.P. Singh (Ed.), *Approximation Theory, Wavelets and Applications*, Kluwer, Netherlands, 1995, pp. 107–130.
- [12] M. Dijić, G. Fongi, A. Maestripieri, The minus order and range additivity, *Linear Algebra Appl.* 531 (2017) 234–256.
- [13] P.A. Fillmore, J.P. Williams, On operator ranges, *Adv. Math.* 7 (1971) 254–281.
- [14] R.E. Hartwig, How to partially order regular elements, *Math. Jpn.* 25 (1) (1980) 1–13.
- [15] C.G. Khatri, S.K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, *SIAM J. Appl. Math.* 31 (1976) 579–585.
- [16] A. Maestripieri, F. Martínez Pería, Decomposition of selfadjoint projections in Krein spaces, *Acta Sci. Math.* 72 (2006) 611–638.
- [17] K.S.S. Nambooripad, The natural partial order on a regular semigroup, *Proc. Edinb. Math. Soc.* (2) 23 (3) (1980) 249–260.
- [18] Heydar Radjavi, James P. Williams, Products of self-adjoint operators, *Michigan Math. J.* 16 (1969) 177–185.
- [19] P. Šmerl, Automorphism of $B(H)$ with respect to minus partial order, *J. Math. Anal. Appl.* 369 (1) (2010) 205–213.